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Some New Periodic Chebycheff Subspaces

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A whole new class of Chebycheff spaces is introduced. The construction starts with a Chebycheff space on the interval $[0, 2\pi]$ and yields a periodic Chebycheff space. By a modification of the construction, derivatives can also be made continuous. Practical and theoretical applications are discussed.

Due to the Remes algorithm, Chebycheff spaces are very desirable objects in numerical analysis. Nevertheless, up to now, only two essentially different kinds of periodic Chebycheff spaces have been known. These are the spaces of trigonometric polynomials and those spaces generated by a particular Krein kernel as given by Forst [2]. Of course, one can induce new spaces by transforming the domain of definition homeomorphically onto itself, but we do not consider these spaces as being essentially different from the original spaces.

We introduce a whole new class of periodic Chebycheff spaces defined on the unit circle K none of which can be obtained from the above spaces by means of a homeomorphic transformation of K onto itself. A special case of this class, that of periodic polynomials, was introduced in [4] in connection with Hermite–Birkhoff interpolation.

These new spaces are not only of theoretical interest., They could be used in place of trigonometric polynomials in the Remes algorithm. Their use would reduce computation time because point evaluations could be done via the Horner scheme.

Let $C(\Omega)$ be the set of real continuous functions on Ω and $C_i(\Omega)$ the set of i-times continuously differentiable functions on Ω. We will be taking $\Omega = K$ and $\Omega = I = [0, 2\pi]$.

An n-dimensional subspace of $C([0, 2\pi])$ or of $C(K)$ is called an n-dimensional Chebycheff subspace if none of its non-zero elements has n or more zeros. A set of n functions $u_1, \ldots, u_n$ is called a Chebycheff system if its span is an n-dimensional Chebycheff subspace. It is called a complete Chebycheff system if $u_1, \ldots, u_r$ is a Chebycheff system for each r, $r = 1, \ldots, n$.

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Starting with a Chebycheff space on $[0, 2\pi]$, we can generate a Chebycheff space on K in the following way.

THEOREM 1. *Let H_n be an n-dimensional Chebycheff subspace of $C[0, 2\pi]$. Let*

$$H_{n,0} = \{h \mid h \in H_n, h(0) = h(2\pi)\}.$$

Then $H_{n,0}$ is an $(n-1)$-dimensional Chebycheff subspace of $D(K)$ if and only if $n-1$ is odd.

Before we prove the theorem, we review some of the properties of the zeros of continuous functions. We use the standard concept of nodal and non-nodal zeros of a continuous function f defined either on the interval $[0, 2\pi]$ or on the circle K. If $t \in K$ and $f(t) = 0$, then we will say that t is a non-nodal zero of f if either $f(t) \geqslant 0$ or $f(t) \leqslant 0$ in some neighborhood of t. Otherwise t is a nodal zero of f. If $t \in (0, 2\pi)$ and $f(t) \leqslant 0$ in some neighborhood of t. Otherwise and also if $t = 0$ or $t = 2\pi$, we say that t is a nodal zero of f.

By $Z_K(f)$, we will denote the number of zeros of f on K counting nodal zeros with multiplicity one and non-nodal zeros with multiplicity two. $Z_I(f)$ is the number of zeros of f on $I = [0, 2\pi]$ counting multiplicities in the same way.

From these definitions, it follows that

(a) $Z_K(f) \leqslant Z_I(f)$ for any $f \in C(K)$

(b) $Z_K(f') \leqslant Z_K(f)$ for any $f \in C_1(K)$

(c) $Z_I(f) - 1 \leqslant Z_I(f')$ for any $f \in C_1([0, 2\pi])$

(d) $Z_K(f)$ is even for any $f \in C(K)$

(e) $Z_I(f) \leqslant n-1$ if f is an element of an n-dimensional Chebycheff subspace of $C([0, 2\pi])$

(f) $Z_K(f) \leqslant n-1$ if f is an element of an n-dimensional Chebycheff subspace of $C(K)$.

Proof of Theorem 1. Let n be even. We first show that $Z_K(h) \leqslant n-2$ for any $h \in H_{n,0}$. Since $h \in H_n$, $Z_I(h) \leqslant n-1$. Thus, using (a) above, we have $Z_K(h) \leqslant n-1$ which is odd. But by (d), $Z_K(h)$ must be even. Therefore $Z_K(h) \leqslant n-2$.

Since $H_{n,0}$ is the subspace of elements of an n-dimensional space which satisfy one linear homogeneous condition, dim $H_{n,0} \geqslant n-1$. This together with $Z_K(h) \leqslant n-2$ for any $h \in H_{n,0}$ implies that $H_{n,0}$ is an $(n-1)$-dimensional Chebycheff subspace of $C(K)$.

Conversely, let V be a Chebycheff subspace of $C(K)$. Then its dimension must be odd. A proof of this fact can be found on p. 180 of [3]. ∎

The basic ideas of Theorem 1 can also be used to generate complete periodic Chebycheff systems in the following situation. Let $w_1, ..., w_n$ be strictly positive elements of $C([0, 2\pi])$. These functions generate a complete n-dimensional Chebycheff system $u_1, ..., u_n$ of $C([0, 2\pi])$ in the standard way [3, 5]:

$$u_1(t) = w_1(t)$$

$$u_2(t) = w_1(t) \int_0^t w_2(s_2)\, ds_2$$

$$\vdots$$

$$u_n(t) = w_1(t) \int_0^t w_2(s_2) \int_0^{s_2} w_3(s_3) \cdots \int_0^{s_{n-1}} w_n(s_n)\, ds_n \cdots ds_2.$$

Let $U_m = \text{span}\{u_1, ..., u_m\}$, $m = 1, ..., n$. Then each U_m is a Chebycheff space.

A second set of spaces we will be using, the reduced subspaces U_m^j, are generated by

$$v_{j,1}(t) = w_{j+1}(t)$$

$$v_{j,2}(t) = w_{j+1}(t) \int_0^t w_{j+2}(s_{j+2})\, ds_{j+2}$$

$$\vdots$$

$$v_{j,m-j}(t) = w_{j+1}(t) \int_0^t w_{j+2}(s_{j+2}) \cdots \int_0^{s_{m-1}} w_m(s_m)\, ds_m \cdots ds_{j+2}.$$

U_m^j is an $(m-j)$-dimensional Chebycheff space for each pair (m, j) with $0 \leqslant j \leqslant m-1$.

These reduced subspaces are analogs of the images of Π_{m-1} (the space of algebraic polynomials of degree not exceeding $n-1$) under D^j (the jth derivative): namely Π_{m-j-1}. In our case, the derivative operator is replaced by D_k which is defined by

$$D_k f(t) = \frac{d}{dt} \frac{f(t)}{w_k(t)} \qquad f \in \mathscr{D}(D_k)$$

and the domain $\mathscr{D}(D_k)$ is

$$\mathscr{D}(D_k) = \{f \mid f \in C([0, 2\pi]),\ f(t)/w_k(t) \in C_1([0, 2\pi])\}.$$

The values $D_k f(0)$ and $D_k f(2\pi)$ are the corresponding one-sided derivatives.

Let $L_k = D_k D_{k-1} \cdots D_1$ with $\mathscr{D}(L_k) = \{h \mid h \in \mathscr{D}(D_1),\ L_j h \in \mathscr{D}(D_{j+1}),\ j = 1, 2, ..., k-1\}$. It follows immediately that $U_n \subset \mathscr{D}(L_k)$ for each k, $0 \leqslant k \leqslant n$. In fact

$$L_k U_m = U_m^k$$

for $k = 0, 1, ..., m-1$, and $L_m U_m = \{0\}$.

The operators D_k can also be defined for periodic functions in the following way:

(a) for $t \in (0, 2\pi)$,

$$D_k f(t) = \frac{d}{dt} \frac{f(t)}{w_k(t)}$$

(b) for $t = 0$ or 2π,

$$D_k f(0) = D^+ \left. \frac{f(t)}{w_k(t)} \right|_{t=0}$$

$$D_k f(2\pi) = D^- \left. \frac{f(t)}{w_k(t)} \right|_{t=2\pi},$$

D^+ and D^- being the right-hand and left-hand derivatives respectively. We take

$$\mathscr{D}(D_k) = \left\{ f \mid f \in C(K),\ \frac{f}{w_k} \in C_1([0, 2\pi]),\ D_k f \in C(K) \right\}.$$

The essential point of this definition is that f and $D_k f$ are required to be periodic functions but not necessarily the intermediate function f/w_k. Whether D_k is considered as an operator on $C([0, 2\pi])$ or $C(K)$ will be clear from the context.

By $\mathcal{N}(L)$, we denote the null-space of the operator L.

If we consider the derivative operator D^k as an operator on $C_k(K)$, a surprising but obvious fact is that $\mathcal{N}(D^k)$ is the space of constant functions. The operators L_k have a similar pleasant property.

Lemma 2. *Let the operators L_k be defined as above with $\mathscr{D}(L_k) \subset C(K)$. If none of the w_i is 2π-periodic, $i = 1, ..., k$, then*

$$\mathcal{N}(L_k) = \{0\}.$$

If at least one of the w_i is 2π-periodic, then

$$\mathcal{N}(K_k) = \text{span}\{g\}$$

for some strictly positive $g \in C(K)$.

Proof. Suppose that $f \in \mathcal{N}(L_k)$. Then, by definition,

$$\frac{d}{dt} \frac{L_{k-1} f(t)}{w_k(t)} = 0$$

and so $L_{k-1} f(t) = c w_k(t)$ for some constant c. If w_k is not periodic, then $c = 0$ since $L_{k-1} f \in C(K)$. We may conclude that if w_k is not periodic, $\mathcal{N}(D_k) = \{0\}$ and so $\mathcal{N}(L_k) = \mathcal{N}(L_{k-1})$. We therefore consider the case when w_k is periodic. Integrating again, we obtain

$$L_{k-2} f(t) = w_{k-1}(t) \left[c \cdot \int_0^t w_k(s)\, ds + d \right]$$

for some constants c, d. As $L_{k-2} f \in C(K)$,

$$w_{k-1}(0) \cdot d = w_{k-1}(2\pi) \left[c \cdot \int_0^{2\pi} w_k(s)\, ds + d \right]$$

or

$$\left[\frac{w_{k-1}(0)}{w_{k-1}(2\pi)} - 1 \right] d = c \cdot \int_0^{2\pi} w_k(s)\, ds.$$

If w_{k-1} is periodic, $c = 0$ for the integral on the right is positive. In this case, $L_{k-2} f(t) = d w_{k-1}(t)$ for some constant d. Thus $L_{k-2} f$ is a multiple of a strictly positive function. If w_{k-1} is not periodic, we can solve for d to obtain

$$\begin{aligned} L_{k-2} f(t) &= c w_{k-1}(t) \left[\int_0^t w_k(s)\, ds + \left[\frac{w_{k-1}(0)}{w_{k-1}(2\pi)} - 1 \right]^{-1} \int_0^{2\pi} w_k(s)\, ds \right] \\ &= c \cdot \frac{w_{k-1}(t)}{w_{k-1}(0) - w_{k-1}(2\pi)} \\ &\quad \times \left[w_{k-1}(0) \int_0^t w_k(s)\, ds + w_{k-1}(2\pi) \int_t^{2\pi} w_k(s)\, ds \right] \end{aligned}$$

for some constant c. The function in brackets is strictly positive (and periodic). Thus again, $L_{k-2} f$ is a multiple of a strictly positive function.

Continuing in this way, we conclude that $\mathcal{N}(L_k) = \{0\}$ if none of the w_i is periodic and $\mathcal{N}(N_k)$ consists of the multiples of some strictly positive function if one of the w_i is periodic. ∎

We are now able to generate a whole family of Chebycheff spaces

$$U_{n,r} = \{ f \in U_n \mid L_k f(0) = L_k f(2\pi),\ k = 0, 1, ..., r \}.$$

We have assumed that $\mathscr{D}(L_k) \subset C([0, 2\pi])$ and have made the convention that $L_0 f(t) = f(t)$.

THEOREM 3. *Let $n-r-1$ be odd. Then $U_{n,r}$ is a Chebycheff subspace of $C(K)$ of dimension $n-r-1$.*

Before we prove the theorem, we need a technical lemma connecting the number of zeros of $D_k f$ with that of f.

LEMMA 4. *Let D_k be considered as an operator on $C(K)$. Then*

$$Z_K(D_k f) \geqslant Z_K(f)$$

for any $f \in \mathscr{D}(D_k)$.

Proof. The conclusion of this lemma does not follow directly from the properties of Z_K listed above because f_k/w_k need not be an element of $C(K)$.

If $f \in \mathcal{N}(D_k)$, then $f \in C(K)$ and $(f/w_k)' \in C(K)$. Let $Z_K(f) = r$ and order the zeros t_i of f so that $0 \leqslant t_0 < t_1 < \cdots < t_p < 2\pi$, where $p \leqslant r$ since some of the zeros may be counted twice. If $t_0 = 0$ and is a nodal zero of f, then f, considered as a function on $I = [0, 2\pi]$ has $Z_I(f) = r+1$. Thus also $Z_I(f/w_k) = r+1$ and so $Z_I((f/w_k)') \geqslant r$. Since these zeros all lie in $(0, 2\pi)$, $Z_K((f/w_k)') \geqslant r$.

Suppose now that $t_0 = 0$ and is a non-nodal zero of f. As above, we can show that $Z_K((f/w_k)') \geqslant r-1$ with all these zeros lying in $(0, 2\pi)$. But 0 is also a zero of $(f/w_k)'$ since it is a nodal zero of f/w_k and since f/w_k is differentiable. Thus, in total, $Z_K(D_k f) \geqslant r$.

The last case is $t_0 > 0$. Then f has r zeros, counting multiplicities, on $(0, 2\pi)$. Thus $(f/w_k)'$ has at least $r-1$ zeros there; i.e., $Z_K(D_k f) \geqslant r-1$. But r is even since $r = Z_K(f)$. Therefore, since $Z_K(D_k f)$ is always even, $Z_K(D_k f) \geqslant r$. ∎

Proof of Theorem 3. We want to show that no non-zero $h \in U_{n,r}$ can have more than $n-r-2$ zeros if $n-r-1$ is odd. Suppose therefore that some $h \in U_{n,r}$ has $n-r-1$ or more zeros. By the definition of $U_{n,r}$, h also belongs to $\mathscr{D}(D_r)$ if L_r is considered as an operator on $C(K)$. Thus we may use Lemma 4 repeatedly to conclude that $Z_K(L_r h) \geqslant n-r-1$. But $Z_K(L_r h)$ must be even. Thus $Z_K(L_r h) \geqslant n-r$. Considering $L_f h$ as a function belonging to $C([0, 2\pi])$, we have $Z_I(L_r h) \geqslant Z_K(L_r h) \geqslant n-r$. But $L_r U_n = U_n^r$, one of the reduced subspaces previously defined, which is an $(n-r)$-dimensional Chebycheff subspace of $C[(0, 2\pi])$. Thus $L_r h$ can only be the zero function, i.e., $L_r h \in \mathcal{N}(L_r)$. By Lemma 2, $h = 0$ or h is a multiple of some strictly positive function. Since h has zeros, $h \equiv 0$ which is the desired contradiction.

$U_{n,r}$ is the space of those elements h of the n-dimensional space U_n which satisfy $r+1$ linear homogeneous conditions $L_k h(0)=L_k h(2\pi)$, $k=0, 1, ..., r$. Therefore $\dim U_{n,r} \geqslant n-(r+1)=n-r-1$. This together with the fact we have just proved that no element of $U_{n,r}$ can have more than $n-r-2$ zeros implies that $\dim U_{n,r}=n-r-1$ and that $U_{n,r}$ is a Chebycheff subspace of $C(K)$. ∎

Our construction yields Chebycheff spaces of functions which, in general, are not differentiable. The same additional assumptions on the w_k which make the basis functions of U_n into an extended complete Chebycheff system system (in the sense of [3, p. 5]) also make the elements of $U_{n,r}$ differentiable on K.

THEOREM 5. *Let $n-r-1$ be odd and $w_k \in C_{r-k+1}(K)$ for* 1, ..., *r. Then* $U_{n,r} \subset C_r(K)$.

Proof. By the definition of $U_{n,r}$, its elements are periodic functions. Karlin and Studden [3] showed that $w_k \in C_{r-k+1}(K)$ implies that $U_{n,r} \subset C_r([0, 2\pi])$. Since any existing derivative of a periodic function is also periodic, it follows that $U_{n,r} \subset C_r(K)$. ∎

If we consider the special case $w_k(t) \equiv 1$, then U_n is the space of algebraic polynomials of degree not exceeding $n-1$. The operators D_k are just ordinary derivatives and $L_k=D^k$. In this case, we denote $U_{n+1,r}$ by $\mathscr{P}_{n,r}$, the space of periodic polynomials. The spaces $\mathscr{P}_{n,r}$ were first introduced in [4].

By combining Theorems 3 and 5, we have

THEOREM 6. *Let $0 \leqslant r \leqslant n-1$ and $n-r$ be odd. Then $\mathscr{P}_{n,r}$ is an $(n-r)$-dimensional Chebycheff subspace of $C_r(K)$.*

A basis for $\mathscr{P}_{n,r}$ for small values of r can be given easily. For example,

$$\mathscr{P}_{n,0}=\operatorname{span}\{1, t(t-2\pi), t^2(t-2\pi), ..., t^{n-1}(t-2\pi)\}, \qquad n \geqslant 2$$

$$\begin{aligned}\mathscr{P}_{n,1}=\operatorname{span}\{&1, t(t-\pi)(t-2\pi),\\ &t^2(t-2\pi)^2, ..., t^{n-2}(t-2\pi)^2\}, \qquad n \geqslant 3\end{aligned}$$

$$\begin{aligned}\mathscr{P}_{n,2}=\operatorname{span}\{&1, t^2(t-2\pi)^2, t^3(t-2\pi)^2-(4/3)\,\pi^2 t(t-\pi)(t-2\pi),\\ &t^3(t-2\pi)^3, ..., t^{n-3}(t-2\pi)^3\}, \qquad n \geqslant 4\end{aligned}$$

$$\begin{aligned}\mathscr{P}_{n,3}=\operatorname{span}\{&1, t^2(t-\pi)(t-2\pi)^2-(4/3)\,\pi^2 t(t-\pi)(t-2\pi),\\ &t^3(t-2\pi)^3-2\pi t^2(t-2\pi)^2, t^3(t-\pi)(t-2\pi)^3,\\ &t^4(t-2\pi)^4, ..., t^{n-4}(t-2\pi)^4\}, \qquad n \geqslant 5.\end{aligned}$$

In general, one can construct $\mathscr{P}_{n,r+1}$ from $\mathscr{P}_{n,r}$. The general pattern is that

$t^{k+1}(t-2\pi)^{r+1} \in \mathscr{P}_{n,r}$ if $k \geqslant r$ and $k+r+2 \geqslant n$. Also $\mathscr{P}_{n,r}$ does not contain any polynomials whose exact degree lies between 1 and $r+1$, inclusively.

These bases for the $\mathscr{P}_{n,r}$ show that they are more efficient in practical applications than trigonometric polynomials as, for example, any $P \in \mathscr{P}_{n,0}$ can be written as

$$P(t) = 1 + t(t-2\pi)\, Q(t),$$

where $Q(t) \in \Pi_{n-2}$. Now $Q(t)$ can be evaluluated efficiently by the Horner scheme. Thus $P(t)$ can be evaluated with less computing time than the corresponding trigonometric polynomial. It is also easy to increase the dimension of the space once the degree r of smoothness has been chosen. One merely adds the basis functions $t^{n-r}(t-2\pi)^{r+1}$ and $t^{n-r+1}(t-2\pi)^{r+1}$ to $\mathscr{P}_{n,r}$ to obtain $\mathscr{P}_{n+2,r}$. The other basis functions need not be recomputed. Finally, the modified Vandermonde determinants involved in interpolation from $\mathscr{P}_{n,r}$ have the same stability as those of polynomial Lagrange interpolation.

The practical usefulness of periodic polynomials was just dwelt on. Their theoretical interpolating power is nevertheless just as good as that of trigonometric polynomials. To justify this claim, we make use of an interpolation theorem of Fitzgerald and Schumaker [1] which we quote in a slightly modified formulation.

THEOREM 7. *Let H_{2n+1} be a $(2n+1)$-dimensional Cebycheff subspace of $C(K)$. Suppose that for any $h \in H_{2n+1}$, $h' \not\equiv 0$ implies that h' cannot have more than $2n$ simple zeros. Let the points $0 = \xi_0 < \xi_{i_1} < \cdots < \xi_{i_k} < 2\pi$ be prescribed, where $J = \{i_0, i_1, ..., i_k\}$ satisfies $0 = i_0 < i_1 < \cdots < i_k < 2n$. If $e = (e_0, e_1, ..., e_{2n})$ is such that $(-1)^i (e_i - e_{i-1}) < 0$, $i = 1, 2, ..., 2n$, and $e_0 > e_{2n}$, then there exists a unique element h of H_{2n+1} and a unique set of points*

$$0 = t_0 < t_1 < \cdots < t_n < 2\pi$$

with

$$t_{i_j} = \xi_{i_j}$$

for $j = 0, 1, ..., k$, such that

$$h(t_i) = e_i$$

for all i, $i = 0, 1, ..., 2n$, and

$$h'(t_i) = 0$$

for all $i \notin J$.

COROLLARY 8. *Let the points* $0 = \xi_{i_0} < \cdots < \xi_{i_k} < 2\pi$ *be prescribed, where* $J = \{i_0, i_1, ..., i_k\}$ *satisfies* $0 = i_0 < i_1 < \cdots < i_k \leqslant 2n$. *If* $e = (e_0, e_1, ..., e_{2n})$ *is such that* $(-1)^i (e_i - e_{i-1}) < 0$, $i = 1, 2, ..., 2n$, *and* $e_0 > e_{2n}$, *then there exists a unique periodic polynomial* $P \in \mathscr{P}_{2n+2,1}$ *and a unique set of points*

$$0 = t_0 < t_1 < \cdots < t_{2n} < 2\pi$$

with

$$t_{i_j} = \xi_{i_j}$$

for $j = 0, 1, ..., k$, *such that*

$$P(t_i) = e_i$$

for all i, $i = 0, 1, ..., 2n$, *and*

$$P'(t_i) = 0$$

for all $i \notin J$.

Proof. We need only show that $\mathscr{P}_{2n+2,1}$ satisfies the assumptions of Theorem 6. But if $P \in \mathscr{P}_{2n+2,1}$, then P is an algebraic polynomial of degree not exceeding $2n+2$, i.e., $P \in \Pi_{2n+2}$. Therefore $P' \in \Pi_{2n+1}$. If $P' \not\equiv 0$, then P' cannot have more than $2n$ zeros.

That $\mathscr{P}_{2n+2,1}$ is a $2n+1$-dimensional Chebycheff space follows from Theorem 6. ∎

In the introduction, we asserted that the spaces constructed here are essentially different from the classical periodic spaces. We show this in one simple case. Let T_1 be the subspace of $C(K)$ spanned by 1, $\sin x$, $\cos x$.

THEOREM 9. *There is no homeomorphism* $g: K \to K$ *for which the mapping* $G: T_1 \to \mathscr{P}_{3,0}$ *given by*

$$T(g(t)) = P(t)$$

for $T \in T_1$, $P \in \mathscr{P}_{3,0}$, *is an isomorphism.*

Proof. We must show that there is no continuous strictly increasing function g mapping $[0, 2\pi]$ onto $[0, 2\pi]$ for which

$$\operatorname{span}\{1, \sin g(t), \cos g(t)\} = \operatorname{span}\{1, t(t-2\pi), t^2(t-2\pi)\}.$$

If this were possible, then one could write

$$\sin g(t) = a_1 + b_1 t(t-2\pi) + c_1 t^2(t-2\pi)$$

$$\cos g(t) = a_2 + b_2 t(t-2\pi) + c_2 t^2(t-2\pi)$$

for some constants a_i, b_i, c_i, $i = 1, 2$.

A tedious but elementary argument (which we omit) which matches the zeros and the extrema of the functions involved shows that this is not possible. ∎

Before concluding, we would like to mention the possibility of using Schumaker's canonical complete Chebycheff systems [5, 6] as a starting point for generating periodic Chebycheff subspaces. Although the operators C_k and L_k are somewhat different (even in the case in which the spaces generated coincide), results similar to Lemma 2 and Theorem 3 can be obtained.

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References

1. C. H. Fitzgerald and L. L. Schumaker, A differential equations approach to interpolation at extremal points, *J. Analyse Math.* **22** (1969), 117–134.
2. W. Forst, Variationsmindernde Eigenschaften eines speziellen Kreinischen Kernes, *Math. Z.* **148** (1976), 67–70.
3. S. Karlin and W. J. Studden, "Tchebycheff Systems: With Applications in Analysis and Statistics," Interscience, New York, 1966.
4. R. A. Lorentz, Simultaneous approximation and Birkhoff interpolation. II: The periodic case, *J. Approx. Theory* **44** (1985), 21–29.
5. L. L. Schumaker, On Tchebycheffian spline functions, *J. Approx. Theory* **18** (1976), 278–303.
6. L. L. Schumaker, "Spline Functions: Basic Theory," Wiley, New York, 1981.